

On the Equitability of Progressive Taxation*

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We propose the principle of equal sacrifice to determine the class of “vertically inequitable” progressive taxes. A necessary condition for an income tax function to be equal sacrifice is formulated, and hence, a subclass of progressive taxes which cannot inflict the same sacrifice upon all individuals relative to *any* strictly increasing and concave utility function is determined. Conversely, it is shown in a general framework that any convex (thus progressive) tax function satisfies the principle of equal sacrifice. Our findings point to the fact that equal sacrifice under progressive income taxation depends heavily upon the degree of marginal rate (as opposed to average rate) progressivity. *Journal of Economic Literature* Classification Numbers: D63, H20. © 1997 Academic Press

1. INTRODUCTION

An important criterion of redistributive justice formulated in the realm of income taxation is the following:

An income tax function should decrease income inequality (in the sense of relative Lorenz dominance) for any given pre-tax income distribution.

This criterion is typically referred to as the “*principle of progressivity*,” for it is now well-known that an income tax function satisfies this principle if, and only if, it is a progressive tax (that is, its average tax rate is

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increasing).¹ This observation may very well account for the striking fact that almost all countries (and certainly all OECD countries) use (statutory) progressive income tax schemes. But a fundamental question remains: *are all progressive taxes equitable?*

A reformulation of John Stuart Mill's famous maxim of income "taxation so as to inflict equal sacrifice" leads us to another compelling redistributive justice principle:

An income tax function must yield equal sacrifice to all individuals relative to at least one acceptable social norm (or, a utility function for the representative agent of the society).

Following Young [31], we call this maxim "*the principle of equal sacrifice*" and contend that it is a useful fairness criterion. True, it is by no means sufficient for determining "*equitable*" taxes perforce, for such a determination can only be done relative to the "actual" social norm that summarizes the preferences of the society, or even better, relative to the "true" preferences of the individuals. But the principle is certainly very effective in elucidating "*inequitable*" taxes, for, by implication, an income tax function not satisfying the principle of equal sacrifice guarantees *unequal* sacrifice relative to *any* possible social norm (utility function for income), and thus in particular, relative to the true preferences of the constituents of the society.²

In this note, we attempt to understand the equity properties of progressive income taxes in the light of the principle of equal sacrifice (or put differently, from the perspective of the doctrine of "*ability to pay*" (cf. Musgrave [21]). An immediate question is then the following: *do all progressive tax functions satisfy the principle of equal sacrifice?* We show that the answer is negative by determining a subclass of progressive taxes which fail to satisfy the principle. Roughly speaking, progressive taxes which are "sufficiently non-convex on a neighborhood" *cannot* yield equal sacrifice for *any* concave and strictly increasing utility function.³ This result illustrates that there is merit in combining the principle of progressivity with the principle of equal sacrifice to pave way towards a theory of equitable income taxation.

The next question is, of course, whether or not the principles of progressivity and equal sacrifice are compatible. We find that they are, and establish

¹ See, for instance, Jakobsson [13], Fellman [10], Eichhorn *et al.* [7], and Lambert [16].

² There is now a small literature on the various aspects of equal sacrifice income taxation: see Richter [25], Buchholz *et al.* [4], Young [30, 31, 32], Yaari [29], Berliant and Gouveia [1], Ok [23], and Mitra and Ok [19].

³ This observation is by no means inconsequential from a practical point of view. For example, one can check that Turkish (statutory) personal income tax was "sufficiently non-convex" around TL 25,000,000 between 1981 and 1985 to guarantee (by Theorem 1) that it was a progressive but not an equal sacrifice income tax (cf. [22, p. 286]). See also Example 1.

that all *convex* progressive taxes do satisfy the principle of equal sacrifice. These results show that equal sacrifice under progressive personal income taxation depends heavily upon the degree of *marginal rate progressivity* (as opposed to the more conventional average rate progressivity).⁴

Our findings extend those reported in Mitra and Ok [19] where it is shown that marginal rate progressivity of tax functions and the equal sacrifice property are essentially equivalent provided that the tax functions are defined piecewise linearly and that the utility function of the representative agent for income is differentiable near the origin. Unfortunately, these two assumptions turn out to be excessively demanding, and in fact, they may well be the real source of Mitra and Ok's characterization of the equal sacrifice tax functions. The present results illustrate that the basic message of [19], however, carries over to a very general framework where neither of these undesirable assumptions are adopted: *a progressive income tax need not be equal sacrifice; one rather needs the marginal rate progressivity to ensure that equal sacrifices will be imposed upon all* (relative to a permissible utility function).

On the other hand, the (almost everywhere) equivalence of marginal progressivity of tax functions and the equal sacrifice property need not hold when we relax the structural assumptions of Mitra and Ok [19]. Indeed, our present results remain silent with respect to a particular subclass of tax functions, roughly speaking that of "mildly non-convex" tax functions. Moreover, we show here that this subclass contains both kinds of the tax functions; those that satisfy the principle of equal sacrifice and those that do not. Of course, finding out exactly which members of this set are actually equal sacrifice tax functions is of interest, for only then a full characterization of non-equitable progressive taxes will be achieved. However, this problem remains open for the moment.

The organization of the paper is described next. In Section 2 we provide precise formulations of the key concepts of the present note. Section 3 states and discusses our main results. It is in this section that we determine some useful subclasses of the sets of progressive equal sacrifice and progressive unequal sacrifice post-tax functions. The above mentioned open question is also formally put forth in this section. In Section 4, we discuss the robustness of our results and find that they are not tight with respect to the relaxation of technical hypotheses. Potential extensions of our findings are also pointed out in this section by means of several examples. The final section supplies the proofs of our main results.

⁴ 1991 U.S. Federal (Statutory) Income Tax is, therefore, found to be respecting the principle of equal sacrifice by virtue of its marginal rate progressivity (see footnote 7). In fact, by the same token, 1979–89 federal *effective* income tax functions estimated by Gouveia and Strauss [11] are all equal sacrifice. (See, however, Mitra and Ok [19] for markedly different conclusions with regard to 1988–90 federal *statutory* income taxes.)

2. PRELIMINARIES

By a *post-tax function*, we mean a continuous, right differentiable and surjective function $f: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ (that associates to pre-tax income x a post-tax income $f(x)$) such that the following conditions hold:

(A1) $f(0) = 0$ and $0 < f(x) < x$ for all $x > 0$,

(A2) $0 < f'_+(x) < 1$ for all $x \geq 0$, where $f'_+(x)$ denotes the right derivative of f at x ,

(A3) $x \mapsto f(x)/x$ is a Lipschitz continuous mapping near the origin; that is, there exists $(y, K) \in \mathbf{R}_{++}^2$ such that

$$\left| \frac{f(x)}{x} - f'_+(0) \right| \leq Kx \quad \text{for all } x \in (0, y].$$

The set of all post-tax functions are denoted by \mathcal{F} . (Notice that, given a post-tax function $f \in \mathcal{F}$, the tax liability levied on income level $x > 0$ is $t(x) \equiv x - f(x)$.)

(A1) is a fairly standard assumption positing that zero income earners do not pay any taxes and that if one earns a positive income, he/she has to pay a positive amount of taxes which must be less than his/her taxable income.⁵ (A2) is also quite standard and assures that a higher income earner pays a higher level of taxes than a lower income earner and that the ranking of taxpayers by pre-tax income and post-tax income is the same. (In other words, by virtue of (A2), we focus only on *non-confiscatory* taxation schemes. Such tax functions are sometimes referred to as *incentive preserving* in the literature (cf. Fei [8], Eichhorn *et al.* [7] and Ok [23]).⁶)

In the literature on income taxation, analyses are typically conducted in terms of differentiable tax functions. Although there is nothing wrong with the differentiability assumption, it clearly makes it difficult to relate the study to the actual taxation practice since the statutory income taxes are typically designed as continuous piecewise linear functions.⁷ On the other

⁵ Since a post-tax function f such that $f(0) > 0$ does not impose any sacrifice on some members of the society, and hence, since it cannot then impose equal sacrifice upon all, as also in Young [30, 31], negative income taxation is excluded from the present study. Consequently, our analysis is conducted solely in terms of statutory income (post-)tax functions.

⁶ We note that some authors identify the notion of *horizontal equity* with that of incentive preservation (cf. Feldstein [9], Plotnick [24] and King [14]). Yet, as argued by Berliant and Strauss [2], such a definition of horizontal equity is not uncontroversial.

⁷ For example, 1991 U.S. Federal Income Tax for single persons was of the following form:

$$t(x) \equiv \begin{cases} 0.15x, & 0 \leq x < 20250 \\ 0.28x - 2632.5, & 20250 \leq x < 49300. \\ 0.31x - 4111.5, & 49300 \leq x \end{cases}$$

The associated post-tax function is of course given by $f(x) = x - t(x)$ for all $x \geq 0$.

hand, if one concentrates only on continuous piecewise linear tax functions, then relating the analysis to the existing literature on income taxation becomes a problem. By assuming only continuity and right differentiability of f and (A3), our framework covers both smooth tax functions and continuous piecewise linear tax functions as special cases. Therefore, although they are a bit tedious to state, these assumptions should be viewed as weak regularity conditions which allow for a definitive generality of analysis.⁸

A post-tax function $f \in \mathcal{F}$ is said to be *progressive* if the average post-tax function $x \mapsto f(x)/x$ is decreasing (i.e., the average tax rate $t(x)/x$ is increasing). One can easily show that a concave (*marginal rate progressive*) post-tax function (i.e., a convex tax function) is progressive but the converse statement does not hold.⁹ We shall denote the class of all progressive post-tax functions by $\mathcal{F}^{\text{prog}}$.

By an *equal sacrifice* post-tax function, we mean a post-tax function $f \in \mathcal{F}$ such that

$$\exists c > 0: [\forall x > 0: [u(x) - u(f(x)) = c]] \quad (1)$$

holds for at least one concave and strictly increasing utility function $u: \mathbf{R}_{++} \rightarrow \mathbf{R}$.^{10,11} This definition is identical to that of Young [31] except that Young requires (1) to be satisfied by a continuous and strictly increasing utility function. But Ok [23] shows that *any* $f \in \mathcal{F}$ is, in fact, an equal sacrifice post-tax function with Young's definition. Therefore, demanding the concavity of the utility function of the individuals is essential to the theory (cf. Mitra and Ok [19]). Moreover, the assumption of

⁸ We shall, in fact, later demonstrate that (A3) is not a necessary condition for our results to hold.

⁹ Define $f \in \mathcal{F}$ as

$$f(x) \equiv \begin{cases} 3x/4, & 0 \leq x < 1 \\ (x/4) + 1/2, & 1 \leq x < 2. \\ x/2, & 2 \leq x \end{cases}$$

One can easily check that $x \mapsto f(x)/x$ defines an everywhere decreasing mapping while f is not concave around 2.

¹⁰ One can easily show that (1) holds for some concave and strictly increasing $u: \mathbf{R}_{++} \rightarrow \mathbf{R}$ if and only if

$$\exists c > 0: [\forall x > 0: [v(x) = cv(f(x))]]$$

holds for some concave and strictly increasing $v: \mathbf{R}_{++} \rightarrow \mathbf{R}$. Therefore, an equal sacrifice post-tax function can be thought of as both an equal *absolute* sacrifice and an equal *proportional* sacrifice post-tax function.

¹¹ As noted by Young [32], this definition of equal sacrifice taxation is best interpreted by considering u as standing for the preferences of a *representative agent* of the society, and thus acting as a *social norm*. Of course, this interpretation saves the principle of equal sacrifice from necessitating interpersonal utility comparisons.

decreasing marginal utility is almost exclusively made in the related public finance literature. By virtue of the usual arguments favoring risk averse behavior, we feel that it is a well-justified assumption.

We should emphasize that, given an equal sacrifice post-tax function, all we know is the existence of a well-behaved utility function relative to which everyone sacrifices equally. Since this utility function may not be a good approximation of the agents' true preferences for income, one cannot conclude that an equal sacrifice post-tax function is, in fact, *vertically equitable*. However, if a post-tax function is *not* equal sacrifice, then we can infer that it cannot inflict the same sacrifice upon everyone relative to *any* sensible utility function. It follows that there is a clear sense in which such taxes are *vertically inequitable*. Therefore, the principle of equal sacrifice is not an inclusion principle identifying the equitable taxes, but is an *exclusion* principle determining the inequitable taxes from the perspective of ability to pay doctrine.

Finally, let us note that our present inquiry is exclusively equity based; in other words, the efficiency aspects of taxation are completely ignored. In particular, we have so far given no reference to the important issue of tax revenue requirement. The usefulness of the equal sacrifice principle may then appear suspect, for, one may argue, an equal sacrifice tax may raise significantly inadequate levels of tax revenue. This argument is, however, misleading since, given any equal sacrifice post-tax function and a feasible level of tax revenue, we may obtain another equal sacrifice post-tax function which raises *at least* this prespecified level of revenue.¹² Consequently, the criticism that the principle of equal sacrifice is not relevant in practice for it is defined independently of the level of tax revenues is, in fact, unwarranted.¹³

¹² CLAIM. Let $f \in \mathcal{F}$ be an equal sacrifice post-tax function, and let $x \in \mathbf{R}_+^m$ be any income distribution. For any fixed tax revenue $0 < R < \sum_{i=1}^m x_i$, there exists an equal sacrifice post-tax function $g \in \mathcal{F}$ such that $\sum_{i=1}^m (x_i - g(x_i)) \geq R$.

Proof. Since, by (A1), it is obvious that $\lim_{n \rightarrow \infty} f^n(x) = 0$ for all $x \geq 0$, the claim will be proved if we can show that f^n is an equal sacrifice post-tax function for all $n \in \{1, 2, \dots\}$. (See footnote 20.) But that $f^n \in \mathcal{F}$ is obvious, and if (1) holds for f , one can easily show by induction that $u(x) - u(f^n(x)) = nc$ so that (1) also holds for f^n . ■

¹³ In addition, one may construct a theory of equal sacrifice taxation in a setting where the level of tax revenue to be raised is exogenously fixed. In fact, this is the traditional framework, and Young [31]'s characterization of equal sacrifice tax methods is obtained precisely in this sort of a setting. (See also [5] and [6].) Nevertheless, as also discussed in [23], in many cases where the actual income distribution is not known prior to taxation, it may not be possible to design a tax policy with a fixed revenue requirement. In such instances, it might be plausible to outline rather a minimum acceptable level of tax revenue along with an estimate of the future income distribution, and as the previous footnote demonstrates, this may well be achieved by equal sacrifice (post-)tax functions.

3. RESULTS

There seems to be a consensus that the concept of progressive taxation carries a considerable degree of egalitarianism with it. Almost all countries use progressive (statutory) taxation schemes and this widespread usage is usually justified on the basis of income inequality aversion. (See [16] for an extensive survey.) Indeed, it is well known that a progressive post-tax function maps a pre-tax income distribution to a more equal post-tax distribution (in the sense of relative Lorenz dominance). Therefore, all progressive post-tax functions are inequality reducing, and hence, they all pass a specific test of distributive justice. We propose another test based on the principle of equal sacrifice: the question is whether all progressive post-tax functions are equal sacrifice. If the answer to this question was yes, then one would conclude that the principle of equal sacrifice is a very weak principle in that it is not useful in further refining the broad class of progressive taxes on the basis of redistributive justice. On the other hand, if the answer was no, then this would mean that the principle of equal sacrifice can be effectively used in assessing the normative properties of progressive taxation.

This appears to be a natural way of making use of the principle of equal sacrifice. It seems to us that the reason this question is not at all addressed in the literature is because the analysis of Samuelson [27, p. 227] is usually taken to imply that the principle of equal sacrifice has no selective power.¹⁴ Many authors (and even some textbooks on public finance) appear to indicate that any progressive post-tax function can be equal sacrifice with respect to a strictly increasing and concave utility function u with a relative risk aversion coefficient greater than 1. Our first result identifies a subclass of progressive post-tax functions which are not equal sacrifice, and hence, shows that this contention is unwarranted.

THEOREM 1. *Let $f \in \mathcal{F}^{\text{prog}}$. If there exists $x_0 > y > 0$ such that $y \geq f(x_0)$ and*

$$f'_+(x_0) f'_+(f(x_0)) > f'_+(y),$$

¹⁴ Put precisely, what [27] shows is the following: given a post-tax function $f \in \mathcal{F}$ and a concave and strictly increasing utility function $u: \mathbf{R}_{++} \rightarrow \mathbf{R}$ satisfying (1), f is progressive if and only if

$$\left| \frac{u'_+(x-f(x))(x-f(x))}{u'_+(f(x))f(x)} \right| \geq 1 \quad \text{for all } x > 0.$$

But this observation is far from clarifying under what conditions (1) can be satisfied for a given $f \in \mathcal{F}$. Indeed, [19] presents an extensive discussion to the effect of showing that the functional equation of (1) can prove to be rather demanding depending on the properties of utility functions.

then there does not exist a strictly increasing and concave utility function $u: \mathbf{R}_{++} \rightarrow \mathbf{R}$ such that

$$\exists c > 0: [\forall x > 0: [u(x) - u(f(x)) = c]].^{15}$$

A related result is reported in Mitra and Ok [19]: For almost any non-concave piecewise linear post-tax function, there does not exist a strictly increasing and concave utility function u which is differentiable near origin and which satisfies (1). However, it is pointed out to us that the driving force behind this observation may well be the mathematical incompatibility between the differentiability of u and the non-differentiability of the post-tax function at finitely many points. On the other hand, Theorem 1 shows that the main premise of [19] can be salvaged even in the absence of these restrictive assumptions: a progressive (post-)tax function need not be equal sacrifice.

To deal with the converse of this theorem we need to study the progressive post-tax functions $f \in \mathcal{F}^{\text{prog}}$ such that

$$\forall x > 0: [\forall y \in [f(x), x]: [f'_+(x) f'_+(f(x)) \leq f'_+(y)]]. \quad (2)$$

Unfortunately, condition (2) is not strong enough to guarantee the existence of a strictly increasing and concave utility function u such that (1) holds. However, if we assume a slightly stronger condition than (2), namely that

$$\forall x > 0: [\forall y \in [f(x), x]: [f'_+(x) \leq f'_+(y)]] \quad (3)$$

we obtain a definitive answer:

THEOREM 2. *Let $f \in \mathcal{F}$. If (3) holds, then f is an equal sacrifice post-tax function.*¹⁶

Clearly, Theorems 1 and 2 remain silent with respect to the progressive post-tax functions which satisfy (2) but do not satisfy (3). In the next section, by presenting appropriate examples, we shall show that such post-tax functions may or may not satisfy (1). The characterization of such post-tax functions which satisfy the principle of equal sacrifice (preferably by a set of easily checkable conditions) stands as an open problem at the moment.

¹⁵ The theorem remains intact if we drop the assumptions of continuity and surjectivity of $f(\cdot)$ and (A3); these properties are not used in the proof of Theorem 1 given in Section 5.

¹⁶ We should note that different versions of this theorem are proved in [23] and [19]. The present formulation is, however, substantially more general than the earlier versions, and of course, covers them as special cases.

4. EXAMPLES

Theorems 1 and 2 together give a very practical way of checking if a given post-tax function is equal sacrifice or not.¹⁷ Our first example illustrates the applicability of these results to the actual taxation practice.

EXAMPLE 1. Define

$$f(x; \alpha) \equiv \begin{cases} 0.9x, & 0 \leq x < 80 \\ \alpha x + (72 - 80\alpha), & 80 \leq x < 90 \\ 0.7x + (10\alpha + 9), & 90 \leq x, \end{cases}$$

and notice that for any $\alpha \in (0, 1)$, $f(\cdot; \alpha) \in \mathcal{F}$. Moreover, $f(\cdot; \alpha)$ is a progressive post-tax function if and only if $\alpha \in (0, 0.9)$. Now, one can easily check that if $f(90; \alpha) \geq 80$, then (2) holds. So, to apply Theorem 1, let $f(90; \alpha) = 10\alpha + 72 < 80$; that is, $\alpha < 0.8$. Choose $x_0 = 90$ and notice that, for any $y \in [f(90; \alpha), 90) \subset (80, 90)$, we have $f'_+(90; \alpha) f'_+(f(90; \alpha); \alpha) > f'_+(y; \alpha)$ if and only if $0.9(0.7) = 0.63 > \alpha$. Thus, one concludes that

$$\exists x > 0: [\exists y \in (f(x; \alpha), x): [f'_+(x; \alpha) f'_+(f(x; \alpha); \alpha) \leq f'_+(y; \alpha)]]$$

if and only if $\alpha \in (0, 0.63)$. Therefore, in view of Theorem 1, $f(\cdot; \alpha)$ is a progressive post-tax function which is *not* equal sacrifice as long as $0 < \alpha < 0.63$. Conversely, if $0.7 \leq \alpha < 0.9$, then by Theorem 2, $f(\cdot; \alpha)$ is a progressive tax which inflicts the same sacrifice upon all income levels relative to a strictly increasing and concave utility function. The indeterminacy region for α corresponding to the case where (2) holds but (3) does not, is $[0.63, 0.7)$. As noted above, whether $f(x; \alpha)$ with $0.63 \leq \alpha < 0.7$ is equal sacrifice or not is an open question.¹⁸

In the next two examples, we shall demonstrate that a progressive post-tax function which satisfies (2) but does not satisfy (3) may or may not be an equal sacrifice post-tax function. Therefore, the subclass of $\mathcal{F}^{\text{prog}}$ where

¹⁷ Although our primary focus in this paper is on progressive post-tax functions, we note that Theorems 1 and 2 remain valid if we replace $\mathcal{F}^{\text{prog}}$ by \mathcal{F} in their statements.

¹⁸ More generally, let $f \in \mathcal{F}$ be defined by

$$f(x) \equiv \begin{cases} \alpha_1 x, & x \in [0, b_1) \\ \alpha_2 x + \theta_1, & x \in [b_1, b_2). \\ \alpha_3 x + \theta_2, & x \in [b_2, \infty) \end{cases}$$

Then, the hypothesis of Theorem 1 holds if, and only if, $\alpha_3 b_2 + \theta_2 < b_1$ and $\alpha_2 < \alpha_1 \alpha_3$. Thus, under these conditions one can conclude that f is *not* an equal sacrifice post-tax function. On the other hand, if $\alpha_1 > \alpha_2 > \alpha_3$, then Theorem 2 entails that f is equal sacrifice.

Theorems 1 and 2 are silent contain both equal sacrifice and unequal sacrifice post-tax functions. In other words, the converse of neither Theorem 1 nor Theorem 2 holds true: Example 2 illustrates that a progressive post-tax function that does not satisfy the precedent of Theorem 1 can be unequal sacrifice; and Example 3 shows that a progressive post-tax function can be equal sacrifice without being concave (that is, without satisfying (3)).

EXAMPLE 2 (Lindsey II).¹⁹ Let

$$h(x) \equiv \begin{cases} 0.1, & \text{if } 0 \leq x < 1 \\ 0.04, & \text{if } x \in \bigcup_{k \in \{0, 2, \dots, 98\}} \left[1 + \frac{k}{100}, 1 + \frac{1+k}{100} \right) \\ 0.05, & \text{if } x \in \bigcup_{k \in \{1, 3, \dots, 99\}} \left[1 + \frac{k}{100}, 1 + \frac{1+k}{100} \right) \\ 0.04, & \text{if } 2 \leq x, \end{cases}$$

and define

$$f(x) \equiv \int_0^x h(u) du \quad \text{for all } x \geq 0.$$

It is easy to observe that $f \in \mathcal{F}$. One can also directly verify that, for any $x > 0$, $\int_0^x h(u) du \geq xh(x)$ so that $f \in \mathcal{F}^{\text{prog}}$. Here we find that

$$\max_{x, z > 0} h(x) h(z) < \min_{y > 0} h(y)$$

so that (2) is trivially satisfied (while (3), of course, fails). We claim that f is not an equal sacrifice post-tax function. Assume, by way of contradiction, that (1) holds for some strictly increasing and concave $u: \mathbf{R}_{++} \rightarrow \mathbf{R}$. Then, upon iteration, we must have $u(x) = u(f^n(x)) + nc$, and hence,

$$u'_+(x) = u'_+(f^n(x)) d_n(x) \quad \text{for all } x > 0 \text{ and } n \in \{\dots, -1, 0, 1, \dots\}, \quad (4)$$

where $d_n(x) \equiv (f^n)'_+(x)$ for all $x > 0$ and $n \in \mathbf{Z}$, and where the right differentiability of u follows from its hypothesized concavity.²⁰ Now define

$$s_k \equiv f \left(1 + \frac{k-1}{100} \right) \quad \text{for all } k \in \{1, \dots, 99\}, \quad \text{and} \quad n_k = \begin{cases} 1, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

¹⁹ This interesting example is communicated to us by Professor John Lindsey II; we gratefully acknowledge our debt to him.

²⁰ For any function $\varphi: \mathbf{R}_+ \rightarrow \mathbf{R}$ and any $n \in \{0, 1, \dots\}$, we define the n th iterate of φ as the function $\varphi^n(x) \equiv (\varphi \circ \dots \circ \varphi)(x)$ for all $x > 0$, where the composition operator is applied n times.

Choosing $x \in \{f^{-n_k}(s_k), f^{-n_k}(s_{k+1})\}$ and $n = n_k$ in (4), we obtain, for all $k \in \{1, \dots, 98\}$,

$$u'_+(f^{-n_k}(s_k)) = u'_+(s_k) d_{n_k}(f^{-n_k}(s_k))$$

and

$$u'_+(f^{-n_k}(s_{k+1})) = u'_+(s_{k+1}) d_{n_k}(f^{-n_k}(s_{k+1})).$$

By concavity of u and the fact that $f^{-n_k}(s_{k+1}) > f^{-n_k}(s_k)$, we thus have

$$\frac{d_{n_k}(f^{-n_k}(s_k))}{d_{n_k}(f^{-n_k}(s_{k+1}))} \geq \frac{u'_+(s_{k+1})}{u'_+(s_k)} \quad \text{for all } k \in \{1, \dots, 98\}. \tag{5}$$

On the other hand, choosing first $n = n_{99} - 1$ and $x = f^{-n_{99} + 1}(s_{99})$, and then $n = n_{99}$ and $x = f^{-n_{99}}(s_1)$ in (4), we have $u'_+(f^{-n_{99} + 1}(s_{99})) = u'_+(s_{99}) d_{n_{99} - 1}(f^{-n_{99} + 1}(s_{99}))$, and $u'_+(f^{-n_{99}}(s_1)) = u'_+(s_1) d_{n_{99}}(f^{-n_{99}}(s_1))$, respectively. Thus, by concavity of u and the fact that $f^{-n_{99} + 1}(s_{99}) < f^{-n_{99} + 1}(1) = f^{-n_{99}}(f(1)) = f^{-n_{99}}(s_1)$, we have

$$\frac{d_{n_{99} - 1}(f^{-n_{99} + 1}(s_{99}))}{d_{n_{99}}(f^{-n_{99}}(s_1))} \geq \frac{u'_+(s_1)}{u'_+(s_{99})}.$$

Combining this with (5) yields that

$$\begin{aligned} A &\equiv \left(\prod_{k=1}^{98} \frac{d_{n_k}(f^{-n_k}(s_k))}{d_{n_k}(f^{-n_k}(s_{k+1}))} \right) \frac{d_{n_{99} - 1}(f^{-n_{99} + 1}(s_{99}))}{d_{n_{99}}(f^{-n_{99}}(s_1))} \\ &\geq \left(\prod_{k=1}^{98} \frac{u'_+(s_{k+1})}{u'_+(s_k)} \right) \frac{u'_+(s_1)}{u'_+(s_{99})} = 1. \end{aligned}$$

But by direct computation, $d_{n_k}(f^{-n_k}(s_k))/d_{n_k}(f^{-n_k}(s_{k+1}))$ is found to be equal to $0.04/0.05 = 0.8$ if $k \in \{1, 3, \dots, 97\}$ and 1 if $k \in \{2, 4, \dots, 98\}$ so that $A = (0.8)^{49} (1/0.04) \simeq 0.00044 < 1$, a contradiction. We conclude that f is not an equal sacrifice post-tax function.

EXAMPLE 3. Define

$$g(x) \equiv \begin{cases} 3x/4, & \text{if } 0 \leq x \leq 1 \\ (x/4) + (1/2), & \text{if } 1 < x \leq 2 \\ x/2, & \text{if } 2 < x. \end{cases}$$

It can be easily checked that $g \in \mathcal{F}^{\text{prog}}$. While proving Theorem 2 in Section 5, we shall show that, for any $f \in \mathcal{F}$ and $x \geq 0$, the iteration sequence $f^n(x)/(f'_+(0))^n$ converges in \mathbf{R}_+ and, for any $c > 0$, $u: \mathbf{R}_{++} \rightarrow \mathbf{R}$ defined as

$$u(x) \equiv \frac{-c}{\log f'_+(0)} \log \lim_{n \rightarrow \infty} \frac{f^n(x)}{(f'_+(0))^n} \quad \text{for all } x > 0,$$

satisfies $u(x) - u(f(x)) = c$ for all $x > 0$. Therefore, if we can show that the function $h: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ defined by $h(x) \equiv \lim_{n \rightarrow \infty} (\frac{4}{3})^n g^n(x)$ for $x \geq 0$, is a concave and strictly increasing function, we may conclude that g is, in fact, an equal sacrifice post-tax function. Now, for $n \in \{3, 4, \dots\}$ we have

$$h_n(x) \equiv \left(\frac{4}{3}\right)^n g^n(x)$$

$$= \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ \left(\frac{4}{3}\right) \left(\frac{1}{4}x + \frac{1}{2}\right), & \text{if } 1 < x \leq 2 \\ \left(\frac{4}{3}\right)^2 \left(\frac{1}{4}\left(\frac{x}{2}\right) + \frac{1}{2}\right), & \text{if } 2 < x \leq g^{-1}(2) \\ \dots, & \dots \\ \left(\frac{4}{3}\right)^{n-1} \left(\frac{1}{4}\left(\frac{x}{2^{n-2}}\right) + \frac{1}{2}\right), & \text{if } g^{2-n}(2) < x \leq g^{1-n}(2) \\ \left(\frac{4}{3}\right)^n \left(\frac{1}{4}\left(\frac{x}{2^{n-1}}\right)\right) + \frac{1}{2}, & \text{if } g^{1-n}(2) < x \leq g^{-n}(2) \\ \left(\frac{4}{3}\right)^n \frac{x}{2^n}, & \text{otherwise.} \end{cases}$$

Clearly, h_n is concave on $[0, g^{-n}(2)]$, and $\{g^{-n}(2)\}_{n=1}^{\infty}$ is a strictly increasing sequence such that $\lim_{n \rightarrow \infty} g^{-n}(2) = \lim_{n \rightarrow \infty} 2^{n+1} = \infty$. Therefore, h is concave on $[0, 1] \cup [1, 2] \cup \bigcup_{n=1}^{\infty} [2, g^{-n}(2)] = [0, \infty)$. Since h is concave and $h'_+(x) > 0$ for all $x \in [0, 2]$, we have

$$\frac{h(b) - h(a)}{b - a} \geq h'_+(b) > 0 \quad \text{whenever } 0 \leq a < b \leq 2,$$

and so h is strictly increasing on $[0, 2]$. Let $x_1, x_2 \in g^{-1}([0, 2])$ and $x_2 > x_1$. Then $(x_1, x_2) = (g^{-1}(a), g^{-1}(b))$ for some $a, b \in [0, 2]$ such that $b > a$, and by strict monotonicity of h on $[0, 2]$,

$$\frac{3}{4}h(x_2) = \frac{3}{4}h(g^{-1}(b)) = h(b) > h(a) = \frac{3}{4}h(g^{-1}(a)) = \frac{3}{4}h(x_1)$$

so we can conclude that h is strictly increasing on $g^{-1}([0, 2])$. But then, by an easy induction argument, it follows that h must be strictly increasing

on $\lim_{n \rightarrow \infty} g^{-n}([0, 2]) = [0, \infty)$. We conclude that g is an equal sacrifice post-tax function.²¹

Our final example is about the robustness of our findings with respect to the boundary condition (A3). One can easily check that (A3) is not necessary for Theorem 1. Indeed, the proof of Theorem 1 given in the next section makes no reference to this assumption. On the other hand, since this assumption is used crucially in the proof of Theorem 2, it is not at all clear if it is necessary for this result. The following example shows that it is not. Therefore, Theorem 2 is also not tight with respect to the relaxation of (A3).

EXAMPLE 4. Let $\beta \in (0, 1)$ and define

$$f(x) \equiv \begin{cases} \beta x - (\beta x^{3/2}/2), & 0 \leq x < 1 \\ \beta x/(x+1), & 1 \leq x. \end{cases}$$

Then, f is a differentiable, surjective and concave function which satisfies (A1) and (A2). For any $K > 0$, there exists $x_0 > 0$ such that $|\beta - f(x)/x| = \beta x/2 \sqrt{x} > Kx$ for all $x \in (0, x_0)$, so f does not satisfy (A3). We claim that f satisfies (1), however. Let us first note that if $\lim_{n \rightarrow \infty} f^n(x)/\beta^n \in (0, \infty)$ then, for any $c > 0$,

$$u(x) \equiv \left(\frac{-c}{\log \beta} \right) \log \lim_{n \rightarrow \infty} \frac{f^n(x)}{\beta^n} \quad \text{for all } x > 0$$

defines a concave and strictly increasing utility function such that $u(x) - u(f(x)) = c$ for all $x > 0$. (The detailed proof of this assertion is given in the next section.) Therefore, all we have to show to conclude that f is equal sacrifice is that $h(x) > 0$ for all $x > 0$, where $h: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is defined by $h(x) \equiv \lim_{n \rightarrow \infty} f^n(x)/\beta^n$. By strict concavity of f , $x \mapsto f(x)/x$ is strictly decreasing, and this implies that $\beta > \dots > f^{n+1}(x)/f^n(x) > \dots > f(x)/x$ for any $x > 0$ and positive integer n . Therefore,

$$h(x) = \prod_{n=0}^{\infty} f^{n+1}(x) x / \beta f^n(x) \in [0, \infty) \quad \text{for all } x > 0$$

²¹ This example establishes that a progressive but not concave post-tax function may be equal sacrifice. In fact, even a convex post-tax function can be equal sacrifice; in a private communication, Professor John Lindsey II showed that the post-tax function $f(x) \equiv \frac{1}{2}(x - a \log(1+x))$ with $a \in (1, 1/10)$ is equal sacrifice with respect to a concave and strictly increasing utility function. See, however, [19].

so that all we have to check is that $h(x) > 0$ for all $x > 0$. Now, for any $x > 0$, $h(x) > 0$ if and only if $\sum_{n=0}^{\infty} (1 - f^{n+1}(x)/\beta f^n(x))$ converges (cf. Theorem 4 of Knopp [15], p. 220). But

$$\sum_{n=0}^{\infty} \left(1 - \frac{f^{n+1}(x)}{\beta f^n(x)} \right) = \frac{1}{2} \sum_{n=0}^{\infty} \sqrt{f^n(x)} \quad \text{for all } x \in [0, 1],$$

so that choosing $x = 1$ and noting that

$$\sqrt{f^n(1)} \leq (\sqrt{\beta})^n / \sqrt{2} \quad \text{for all } n \in \{0, 1, \dots\},$$

we learn that $\sum_{n=0}^{\infty} \sqrt{f^n(1)} < \infty$, and therefore $h(1) > 0$. By the monotonicity of f , h is a non-decreasing function, and hence by the previous observation, we must have $h(x) > 0$ for all $x \geq 1$. Finally, since h is a concave function (being the limit of a convergent sequence of concave functions) and $h(0) = 0$, we have $h(x)/x \geq h(1)$ for all $x \in (0, 1)$, so that $h(x) > 0$ for all $x \in (0, 1)$ as well, and the claim is proved.²²

5. PROOFS

Proof of Theorem 1. Assume the hypotheses of the theorem, and let

$$u(x) - u(f(x)) = c \quad \text{for all } x > 0,$$

for some $c > 0$ and $u: \mathbf{R}_{++} \rightarrow \mathbf{R}$ which is strictly increasing and concave. Then, for any $x > 0$ and $\varepsilon > 0$,

$$u(x + \varepsilon) - u(f(x + \varepsilon)) = u(x) - u(f(x)).$$

Pick an arbitrary $z > 0$. Then, since $f'_+(z) > 0$, there is $\bar{\varepsilon} > 0$ such that for all $0 < \varepsilon < \bar{\varepsilon}$, we have $f(z + \varepsilon) - f(z) > 0$ (cf. Graves [12, Theorem 2, p. 70]). But then we can write

²² In effect, this example establishes the fact that (A3) is not necessary for $\lim_{n \rightarrow \infty} f^n(x)/\beta^n \in (0, \infty)$ for all $x > 0$. The following result, due to Seneta [28], gives a necessary and sufficient condition for this convergence to hold: $\lim_{n \rightarrow \infty} f^n(x)/\beta^n \in (0, \infty)$ for all $x > 0$ if, and only if,

$$\int_1^{\infty} \left(\frac{\beta}{a} - f\left(\frac{1}{a}\right) \right) da < \infty.$$

(In Example 4, for instance, we have $\int_1^{\infty} (\beta/a - f(1/a)) da = \beta$.) (A3), of course, implies this integral condition, but not conversely.

$$\begin{aligned} u(z + \varepsilon) - u(z) &= u(f(z + \varepsilon)) - u(f(z)) \\ &= (f(z + \varepsilon) - f(z)) \left(\frac{u(f(z) + (f(z + \varepsilon) - f(z))) - u(f(z))}{f(z + \varepsilon) - f(z)} \right) \end{aligned}$$

for $0 < \varepsilon < \bar{\varepsilon}$. Thus, for all $0 < \varepsilon < \bar{\varepsilon}$,

$$\frac{u(z + \varepsilon) - u(z)}{\varepsilon} = \left(\frac{f(z + \varepsilon) - f(z)}{\varepsilon} \right) \left(\frac{u(f(z) + (f(z + \varepsilon) - f(z))) - u(f(z))}{f(z + \varepsilon) - f(z)} \right).$$

Letting $\varepsilon \rightarrow 0$, and noting that $f(z + \varepsilon) - f(z) > 0$ for $0 < \varepsilon < \bar{\varepsilon}$, we obtain

$$u'_+(z) = f'_+(z) u'_+(f(z)) \quad \text{for all } z > 0. \quad (6)$$

Now, since $x_0 > y$ and u is concave, $u'_+(y) \geq u'_+(x_0)$, and (6) yields

$$u'_+(f(y)) \geq \left(\frac{f'_+(x_0)}{f'_+(y)} \right) u'_+(f(x_0)). \quad (7)$$

Letting $z = f(x_0)$ in (6), $u'_+(f(x_0)) = f'_+(f(x_0)) u'_+(f^2(x_0))$ and so (7) gives

$$u'_+(f(y)) \geq \left(\frac{f'_+(x_0) f'_+(f(x_0))}{f'_+(y)} \right) u'_+(f^2(x_0)). \quad (8)$$

But since $f(x_0) < y$ and f is strictly increasing, $f^2(x_0) < f(y)$ and by concavity of u , $u'_+(f^2(x_0)) \geq u'_+(f(y))$. Therefore, (8) gives

$$u'_+(f^2(x_0)) \geq \left(\frac{f'_+(x_0) f'_+(f(x_0))}{f'_+(y)} \right) u'_+(f^2(x_0))$$

and this contradicts the hypothesis that $f'_+(y) < f'_+(x_0) f'_+(f(x_0))$. ■

Proof of Theorem 2. Let $f \in \mathcal{F}$ and assume that (3) holds. Suppose that there exist $x_0 > 0$ and $y > 0$ such that $x_0 > y$ and $f'_+(x_0) > f'_+(y)$. Since $0 < f(x_0) < x_0$, we have $\lim_{n \rightarrow \infty} f^n(x_0) = 0$, and therefore, there must exist a positive integer n_0 such that $y \in [f^{n_0+1}(x_0), f^{n_0}(x_0)]$. (Here $f^0(x) = x$, and for any $n \geq 1$, f^n is the n th iterate of f ; see footnote 20.) Applying (3) at $x = f^{n_0}(x_0)$, we then have

$$f'_+(f^{n_0}(x_0)) \leq f'_+(y) < f'_+(x_0).$$

But this is impossible, for by applying (3) successively,

$$f'_+(x_0) \leq f'_+(f(x_0)) \leq f'_+(f^2(x_0)) \leq \dots \leq f'_+(f^{n_0}(x_0)).$$

We therefore conclude that $f'_+(x) \leq f'_+(y)$ whenever $0 < y < x$; that is, f'_+ is decreasing on $(0, \infty)$. Since f is continuous, we can apply Proposition 18 of Royden [26, p. 114], to conclude that f is concave on $(0, \infty)$.

Now let $f'_+(0) = \beta$ and define, for any $x > 0$,

$$h_n(x) \equiv \frac{f^n(x)}{\beta^n} \quad \text{for all } n \in \{0, 1, \dots\} \quad \text{and} \quad h(x) \equiv \lim_{n \rightarrow \infty} h_n(x).$$

Assume for the moment that $h(x) \in (0, \infty)$ for all $x > 0$, and, for any $c > 0$, define the function $u: \mathbf{R}_{++} \rightarrow \mathbf{R}$ as

$$u(x) \equiv \frac{-c}{\log \beta} \log h(x) \quad \text{for all } x > 0.$$

Note that h is the limit function of a convergent sequence of concave functions, and hence, it is concave on $(0, \infty)$. This implies that u is a concave function. Moreover, for all $x > 0$,

$$\begin{aligned} \frac{-c}{\log \beta} \log \lim_{n \rightarrow \infty} \frac{f^{n+1}(x)}{\beta^n} &= \frac{-c}{\log \beta} \log \left(\beta \lim_{n \rightarrow \infty} \frac{f^{n+1}(x)}{\beta^n} \right) \\ &= \frac{-c}{\log \beta} \log \lim_{n \rightarrow \infty} \frac{f^n(x)}{\beta^n} - c, \end{aligned}$$

that is, $u(f(x)) = u(x) - c$.

We now proceed to show that $h(x) \in (0, \infty)$. Let $x > 0$ be arbitrary. We have, for all $n \in \{0, 1, \dots\}$,

$$\frac{f^{n+1}(x)}{\beta^{n+1}} = \frac{f(f^n(x))}{\beta f^n(x)} \frac{f(f^{n-1}(x))}{\beta f^{n-1}(x)} \dots \frac{f(x)}{\beta x} x$$

so that

$$h(x) = \lim_{n \rightarrow \infty} \frac{f^n(x)}{\beta^n} = \left(\prod_{n=0}^{\infty} \frac{f^{n+1}(x)}{\beta f^n(x)} \right) x.$$

Therefore, $h(x) \in (0, \infty)$ if and only if

$$\prod_{n=0}^{\infty} \frac{f^{n+1}(x)}{\beta f^n(x)} \in (0, \infty). \quad (9)$$

Since f is concave, $t \mapsto f(t)/t$ is decreasing. Moreover, $\lim_{t \rightarrow 0} f(t)/t = \beta$, and hence $f^n(x)/\beta f^{n-1}(x) \leq 1$ for each n . Consequently, by Theorem 4 of [15, p. 220], (9) holds if and only if

$$\sum_{n=0}^{\infty} \left(1 - \frac{f^{n+1}(x)}{\beta f^n(x)} \right) \quad (10)$$

is convergent. By (A3) and the fact that $\lim_{n \rightarrow \infty} f^n(x) = 0$, there must exist an integer N and $K > 0$ such that

$$\left| \frac{f(f^n(x))}{f^n(x)} - \beta \right| \leq K f^n(x) \quad \text{whenever } n \geq N$$

and thus,

$$\sum_{n=N}^{\infty} \left(1 - \frac{f^{n+1}(x)}{\beta f^n(x)} \right) \leq \frac{K}{\beta} \sum_{n=N}^{\infty} f^n(x). \quad (11)$$

We shall next show that $\sum_{n=N}^{\infty} f^n(x)$ is convergent. Let $\gamma \in (\beta, 1)$. Notice that since $\{f^{n+1}(x)/f^n(x)\}_{n=0}^{\infty}$ is a sequence converging to β , there exists an integer L such that $n \geq L$ implies $f^{n+1}(x) < \gamma f^n(x)$. But this yields

$$f^{L+\ell}(x) < \gamma^\ell f^L(x) \quad \text{for all } \ell = 1, 2, \dots,$$

and therefore, we have

$$\sum_{n=L}^{\infty} f^n(x) < f^L(x) \sum_{\ell=0}^{\infty} \gamma^\ell < \infty.$$

Combining this observation with (11), we learn that the series in (10) is convergent, which proves (9).

Finally, we show that h is strictly increasing. To this end, note that h is continuous on $(0, \infty)$ since it is concave on $(0, \infty)$. Furthermore, in the process of establishing (9) above, we noted that, given any $x > 0$ and positive integer n , $f^{n+1}(x)/\beta f^n(x) \leq 1$, and so $h_n(x) \leq x$. Thus, for every $x > 0$, $0 \leq h(x) \leq x$ so that h is continuous at 0 as well. Since $h(0) = 0$ and $h(x) > 0$ for $x > 0$, we must have $h'_+(y) > 0$ for some $y > 0$ (cf. [26, Proposition 2, p. 99]). Then, by concavity of h , $h'_+(x) > 0$ for all $x \in (0, y)$; that is h is strictly increasing on $(0, y)$. Now, let $x_1, x_2 \in f^{-1}((0, y))$ and $x_2 > x_1$. Then, $x_1 = f^{-1}(a)$ and $x_2 = f^{-1}(b)$ for some $a, b \in (0, y)$ such that $b > a$. Since h satisfies $h(f(x)) = \beta h(x)$ for all $x \geq 0$, we have $\beta h(f^{-1}(b)) = h(b) > h(a) = \beta h(f^{-1}(a))$ since h is known to be strictly increasing on $(0, y)$. Thus, $h(f^{-1}(b)) = h(x_2) > h(x_1) = h(f^{-1}(a))$ proving that h is strictly increasing on $f^{-1}((0, y))$. By induction, it follows that h is strictly increasing

on $f^{-n}((0, y))$ for all $n \in \{0, 1, \dots\}$ which, in turn, implies that h is strictly increasing on $\bigcup_{n=0}^{\infty} f^{-n}((0, y)) = (0, \lim_{n \rightarrow \infty} f^{-n}(y))$. But by (A2), $\dots > f^{-2}(y) > f^{-1}(y) > y$ so that $\{f^{-n}(y)\}_{n=0}^{\infty}$ is a strictly increasing sequence. If $\{f^{-n}(y)\}_{n=0}^{\infty}$ was bounded, then we would have $\lim_{n \rightarrow \infty} f^{-n}(y) = M$ for some $M > 0$, and by continuity of f^{-1} and (A2), we would obtain the following contradiction:

$$\lim_{n \rightarrow \infty} f^{-(n+1)}(y) = \lim_{n \rightarrow \infty} f^{-1}(f^{-n}(y)) = f^{-1}(\lim_{n \rightarrow \infty} f^{-n}(y)) = f^{-1}(M) > M.$$

Therefore, $\{f^{-n}(y)\}_{n=0}^{\infty}$ cannot be bounded, and we conclude that $\lim_{n \rightarrow \infty} f^{-n}(y) = \infty$. Consequently, h , and thus u , is strictly increasing on $(0, \infty)$. ■

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